Journal of Statistical Physics, Vol. 33, No. 3, 1983

Medium-Dependent Dynamics in Fermi Systems from a Pair-Composite Viewpoint

Abraham Goldberg[1] **and Robert D. Puff**[2]

Received January 25, 1983; revised May 3, 1983

We examine the effects of medium dependence of the two-body dynamics on the many-body properties of Fermion systems, with approximation ultimately aimed at lower densities for all temperatures. The dynamics are initially treated in terms of a pair-composite formulation given previously, and the underlying single-Fermion nature of the pair constituents allows interpretation via more conventional thermal many-body formalism. This permits construction of coupled equations for composite amplitudes and bound states, single-particle energy and momentum distributions, and macroscopic thermodynamic properties. We explore differences between our results and those of traditional theories which incorporate two-body correlations in some fashion, and we display explicitly how correct limiting results are recovered from our equations when the density and/or coupling strength is decreased. Finally, we provide an interpretation of our results via a form of quasiparticle quantum cluster expansion analogous to the familiar particle quantum cluster expansion.

KEY WORDS: Fermi systems; pair-composite correlations.

1. INTRODUCTION

The many-body problem for physically interesting systems is plagued by difficulties, of both mathematical and conceptual nature, sufficiently profound that exact solutions can be found only in certain limits of the

---

This work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore National Laboratory under contract No. W-7405-ENG-48.

[1] Lawrence Livermore National Laboratory, University of California, Livermore, California 94550.

[2] Institute for Nuclear Theory, Physics Department, University of Washington, Seattle, Washington, 98195.

parameters characterizing the system. (Of course, there are model problems which have exact solutions.) For small parameters one can usually expand the thermodynamic properties of the actual system about those of a simpler reference system. The simplest such expansion is perturbation theory, an expansion in the coupling constant of the two-body interaction.(1) Two perhaps more realistic limits for which well-known expansions exist are, first, that of "low" number density n and "high" temperature T and, second, that of "low" density and "low" temperature. Derivation of the expansion applicable in the latter regime, particularly at $T = 0$, was the object of numerous early applications of quantum field theoretic techniques to statistical physics.[3] The series applicable in the former regime is the familiar virial expansion.(3) In both cases the density is sufficiently low that $na^3 < 1$, where a is some length characterizing the potential, and the reference system is the ideal gas. It is not surprising that in the two limits in which the problem is soluble, the condition $na^3 \ll 1$ should be met, for it affords an enormous simplification of the many-body theory. This simplification is that the dynamics of two-body interactions within the many-body medium can be treated as independent of that medium. The scattering and bound states of any two particles can be handled as if these particles were *in vacuo*. The condition $na^3 < 1$ can be said to be that of medium-independent dynamics. The extent to which two-body scattering is affected by the medium and the nature of the corrections with the small na^3 constraint then poses a subject for investigation.

This paper has a twofold purpose. First, we examine in detail the effects of medium dependence on the two-body scattering. The usual approach to this issue, formulated in terms of thermal Green's functions,(4–6) has been via the propagation and scattering of single Fermions in the medium. A second approach, which is an outgrowth of a formulation discussed previously, treats the scattering pair of Fermions as a unit propagating through the medium. This composite unit is taken as the fundamental component of the medium, and the internal dynamics of the composite are manifested through nonstandard (Boson) commutation rules of the composite fields. It is this latter formulation which we use in this paper. We are then led unambiguously to a two-particle T matrix, satisfying a Lippmann–Schwinger equation. In Section 2, we elucidate the structure of this T matrix, and show how it gives two-body bound states and scattering amplitudes which include the effects of the surrounding medium.

There is still the question as to how these microscopic two-body properties affect the macroscopic thermodynamic averages of the many-body system, and it is our second purpose to address this question. This connection is nontrivial even in the limit of medium-independent scatter-

---

[3] For a review and references concerning the $T = 0$ ground state formalism, see Ref. 2.

ing. (See Section 4.) Consequently a second condition (in addition to na^3 being small) has been traditionally imposed. There are two obvious choices for this second condition and they lead to the two different known expansions.

One choice for this second condition is $n\lambda^3 < 1$, where λ is the thermal wavelength $(2\pi\hbar^2/mk_BT)^{1/2}$. This condition, together with $na^3 < 1$, marks the resultant theory as a true "low-density" expansion, since n is smaller than either of the other naturally occurring densities a^{-3} or λ^{-3}. This theory is in fact the virial expansion.(3) The additional requirement that $n\lambda^3 < 1$ indicates that the system is nondegenerate. Consequently, Fermi or Bose statistics do not enter via characteristic distribution functions; only Boltzmann factors appear. However, statistics do enter through exchange effects if $\lambda/a > 1$. If $\lambda/a < 1$, even exchange effects disappear and the virial results are completely independent of statistics. Both $\lambda/a \lessgtr 1$ are compatible with $na^3 < 1$ and $n\lambda^3 < 1$, of course.

Another choice for a second condition on the theory is $\lambda/a > 1$, which gives what may be called a short-range interaction expansion; i.e., a is smaller than either of the other naturally occurring lengths λ and $n^{-1/3}$. In such a theory exchange effects are always important. If the temperature is such that $n\lambda^3 < 1$, the expansion shares a region of applicability with the virial expansion. If, on the other hand, $n\lambda^3 > 1$, statistics enter through both exchange effects and degeneracy factors (i.e., characteristic Fermi or Bose functions). The details of this theory have been carried out only at absolute zero (where both λ/a and $n\lambda^3$ are infinite).(8)

The above discussion is summarized schematically in Fig. 1, where the region of validity in the temperature–density plane of the virial and

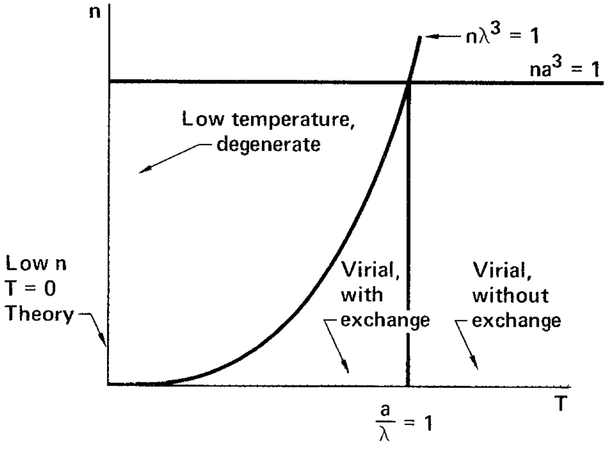

Fig. 1.

zero-temperature theories are shown. There is no overlap between these limiting cases, although a true medium-independent treatment ($na^3 < 1$) must include both. Such a treatment has in the past not received much attention although the necessary theoretical framework is well known. One reason for this situation is that interesting systems like bulk He^3 liquid exist at densities such that the $na^3 < 1$ criterion is simply not applicable, and low-temperature properties must be investigated by other methods. Thus, one may ask if complete treatment of the $na^3 < 1$ regime for all temperatures is applicable to any system or justified by any experimental situation. Indeed such a situation appears to occur for He^3 monolayers adsorbed on graphite, i.e., the monolayers apparently remain in the gaseous phase down to absolute zero (at sufficiently low density). That is, adsorbed He^3, unlike bulk He^3 or He^4, exists in a low-density phase at all temperatures.[4] A low-density expansion applicable to all temperatures would presumably be appropriate for calculating thermodynamic properties of this system. Calculations along these lines have been performed by Vetrovac and Carneiro.(11)

With increasing density, a medium-independent approximation is clearly inadequate. In Section 3 we give a prescription for calculating thermodynamic quantities, in particular the density as a function of temperature and chemical potential, and this in principle determines all thermodynamic quantities. This prescription uses the medium-dependent dynamics, but could be carried to the medium-independent limit. Thus the complete description requires the pressure

$$P(\mu, T) = \int_{-\infty}^{\mu} n(\mu', T)\, d\mu' \tag{1.1}$$

the entropy density

$$S(\mu, T) = \left.\frac{\partial P}{\partial T}\right|_{\mu} \tag{1.2}$$

and finally the energy density

$$E(\mu, T) = \mu n + TS - P \tag{1.3}$$

In order to obtain the density n to generate the macroscopic properties, Eqs. (1.1)–(1.3), one may simply integrate the single-particle momentum distribution $n(\mathbf{k})$. However, for our purposes, this requires establishing the connection between single-particle properties (like the momentum distribution and single-particle energy) and the composite amplitude and bound

---

[4] The experimental situation is described in Ref. 9; a virial expansion analysis can be found in Ref. 10.

states. This is done in Section 3 by retreating to traditional methods to approximate the single-particle properties which appear (according to our *interpretation*) in the composite state equations. The approximations needed to generate our composite amplitudes and bound states by traditional formalism involves certain on-energy-shell replacements in energy denominators at specific points in the calculation. The same prescription then permits approximate determination of the single-particle properties $n(\mathbf{k})$ and $\epsilon(\mathbf{k})$ in a manner consistent with the original pair-composite results. In the language of the single-particle formalism, such an approximation represents a truncation of certain diagrams. As it happens, this truncation is such that no single-particle lifetimes appear explicitly since only the momentum distribution and the real part of the single-particle energy are parameters of the theory. Lifetime effects are clearly included, however, and the single-particle lifetime could be calculated by again appealing to the traditional formalism. The final coupled equations involve the composite amplitudes and bound states, together with the single-particle energy and momentum distribution, for fixed values of the chemical potential and temperature. In the last analysis, once these relationships have been established, with feedback couplings of composite dynamics, particle dynamics, and thermodynamic parameters, the resulting equations are then in any application to be solved self-consistently.

The analysis in Section 3 may be subjected to two distinct limiting cases. Full description of thermodynamic quantities have been given to second order in the coupling constant (perturbation theory). Furthermore, the well-known virial expansion gives the density to second order in the fugacity parameter $\zeta = e^{\beta\mu}$ with β the inverse temperature and μ the chemical potential. This expansion is valid in the limit of low density, where all dynamical quantities become medium independent. We require of any medium-dependent theory that it recapture both these limits. We also expect to recapture the density perturbation theory at zero temperature first obtained by Galitskii.(8)

We also impose on any theory the rather subjective requirement that the momentum distribution be interpretable in terms of single-particle motion through the medium plus additional terms clearly arising from two-body correlations. This interpretation is provided in Section 4 as a quasiparticle quantum cluster expansion.

2. MEDIUM-DEPENDENT TWO-BODY DYNAMICS

The formulation of the many-body problem in terms of composites leads to a hierarchy of composite Green's functions.(7) A factorization approximation for the two-composite propagator, analogous to the ordinary Hartree–Fock factorization for the two-*particle* propagator (i.e., dynamical correlations between *composites* are thus neglected) gives unambiguously a T-matrix satisfying a Lippmann–Schwinger equation

$$T_S(\mathbf{P},z) = V_S + V_S \frac{\mathcal{N}(\mathbf{P})}{z - \mathcal{H}_0(\mathbf{P})} T_S(\mathbf{P},z) \tag{2.1}$$

where **P** and S are the total momentum and total spin of the composite. (While the two-body interaction preserves total spin, it may yet depend on S.) **P** and S are to be regarded as fixed parameters. The noninteracting Hamiltonian $\mathcal{H}_0(\mathbf{P})$ and the "blocking" operator $\mathcal{N}(\mathbf{P})$ are diagonal in the plane wave representation for the internal pairs,

$$\langle \mathbf{k}|\mathcal{H}_0(\mathbf{P})|\mathbf{k}'\rangle = \Omega(\mathbf{P},\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') \tag{2.2}$$

and

$$\langle \mathbf{k}|\mathcal{N}(\mathbf{P})|\mathbf{k}'\rangle = N(\mathbf{P},\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') \tag{2.3}$$

with

$$\Omega(\mathbf{P},\mathbf{k}) = \epsilon(\mathbf{k}_1) + \epsilon(\mathbf{k}_2) \tag{2.4}$$

and

$$N(\mathbf{P},\mathbf{k}) = 1 - n(\mathbf{k}_1) - n(\mathbf{k}_2) \tag{2.5}$$

with

$$\mathbf{k}_1 = \tfrac{1}{2}\mathbf{P} + \mathbf{k}, \qquad \mathbf{k}_2 = \tfrac{1}{2}\mathbf{P} - \mathbf{k}$$

The matrix element $\langle \mathbf{k}|V_S|\mathbf{k}'\rangle$ in this representation is the ordinary Fourier transform of the two-body potential

$$V_S(\mathbf{k}-\mathbf{k}') = \int d^3r\, e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} V_S(\mathbf{r})$$

in spin state S.

These composites embody hidden additional symmetries (i.e., they are in fact formed from Fermion pairs). These symmetries manifest themselves through nonstandard commutation relations for the composite field operators. The nonstandard rules in fact are responsible for the "blocking" operator $N(\mathbf{P},\mathbf{k})$. If the total number of composites is $N/2$, $n(\mathbf{k}_1)$ must be normalized to

$$N = 2\sum_{\mathbf{k}_1} n(\mathbf{k}_1) \tag{2.6}$$

We therefore *interpret* $n(\mathbf{k}_1)$ as a single constituent momentum distribution. Similarly, $\epsilon(\mathbf{k}_1)$ is *interpreted* as the real part of the energy of a single constituent with momentum $\mathbf{k}_1$ in the medium.

Although Eq. (2.1) emerges from the consideration of a many-composite system, we need not adhere to that view here, but rather simply regard Eq. (2.1) as the basic defining equation for the T-matrix operator. This definition, together with the interpretations of $\mathscr{H}_0(\mathbf{P})$ and $\mathscr{N}(\mathbf{P})$, appears quite reasonable, and it should be emphasized that in fact Eqs. (2.1)–(2.5) emerge naturally from the composite formulation.

The traditional approach, given through the two-particle propagator for the Fermion constituents, also leads to various T-matrix theories which can be interpreted in a composite picture.(12) The simplest of these employ single-particle propagators in intermediate states which are either completely free, have the free particle form with correct chemical potential,(13) or have the (infinite lifetime) Hartree–Fock structure. Such an approach leads to a T-matrix equation similar to Eq. (2.1), but with the "blocking" operator replaced by $1 - f_F(\epsilon(\mathbf{k}_1)) - f_F(\epsilon(\mathbf{k}_2))$. To lowest order $\epsilon(\mathbf{k}_1)$ is the Hartree–Fock single-particle energy, although higher-order self-energy terms are frequently included. $f_F(\epsilon(\mathbf{k}_1))$ is replaced at zero temperature by the step function with a proper cutoff at the correct Fermi momentum. This scheme is the finite-temperature version of the Galitskii integral equations(8) (see Section 3).

Further extensions of T-matrix methods, which attempt to include off-energy-shell effects in intermediate states,(14) present the difficulty that no simple analysis via medium-dependent pair wave functions is possible. In our case, such an analysis is indeed quite simple, because our composite theory leads us to *begin* with Eq. (2.1) as the correct structure rather than one of the various possibilities emerging from the Fermion constituent treatments.

Since $\Omega(\mathbf{P}, \mathbf{k})$ is real, $T_S(\mathbf{P}, z)$ is analytic for all finite z off the real axis. For a given total momentum $\mathbf{P}$, $\Omega(\mathbf{P}, \mathbf{k})$ has a minimum value $\Omega_0 = \Omega(\mathbf{P}, 0)$, and $T_S(\mathbf{P}, z)$ has a branch cut for z real, $z > \Omega_0$, corresponding to the constituents scattering in the medium. In addition, the two constituents may possess bound states at energies $E_\nu(\mathbf{P}, S) < \Omega_0$, giving $T_S(\mathbf{P}, z)$ simple poles at $z = E_\nu(\mathbf{P}, S)$. We then have

$$\langle\mathbf{k}|T_S(\mathbf{P}, z)|\mathbf{k}'\rangle = \langle\mathbf{k}|\tilde{T}_S(\mathbf{P}, z)|\mathbf{k}'\rangle + \sum_\nu \frac{\langle\mathbf{k}|F_\nu(\mathbf{P}, S)|\mathbf{k}'\rangle}{z - E_\nu(\mathbf{P}, S)} \tag{2.7}$$

with $\tilde{T}_S(\mathbf{P}, z)$ analytic but for the branch cut. $\langle\mathbf{k}|F_\nu(\mathbf{P}, S)|\mathbf{k}'\rangle$, the residue at the νth pole, is related to the bound state wave function.

A dispersion relation for $T_S(\mathbf{P}, z)$ follows from these analytic properties

$$\langle\mathbf{k}|T_S(\mathbf{P}, z)|\mathbf{k}'\rangle = \langle\mathbf{k}|T_S(\mathbf{P}, z')|\mathbf{k}'\rangle + \int_{-\infty}^{\infty} d\omega\, \langle\mathbf{k}|\Gamma_S(\mathbf{P}, \omega)|\mathbf{k}'\rangle \left(\frac{1}{z-\omega} - \frac{1}{z'-\omega}\right) \tag{2.8}$$

with the real, positive definite spectral function $\Gamma_S(\mathbf{P}, \omega)$ given by

$$\begin{aligned} &\langle\mathbf{k}|\Gamma_S(\mathbf{P}, \omega)|\mathbf{k}'\rangle \\ &\quad = \frac{i}{2\pi}\left\{\langle\mathbf{k}|T_S(\mathbf{P}, \omega + i\eta)|\mathbf{k}'\rangle - \langle\mathbf{k}|T_S(\mathbf{P}, \omega - i\eta)|\mathbf{k}'\rangle\right\} \\ &\quad = \sum_\nu \langle\mathbf{k}|F_\nu(\mathbf{P}, S)|\mathbf{k}'\rangle\delta(\omega - E_\nu(\mathbf{P}, S)) \\ &\qquad + \frac{i}{2\pi}\theta(\omega - \Omega_0)\lim_{\eta\to 0}\left\{\langle\mathbf{k}|\tilde{T}_S(\mathbf{P}, \omega + i\eta)|\mathbf{k}'\rangle - \langle\mathbf{k}|\tilde{T}_S(\mathbf{P}, \omega - i\eta)|\mathbf{k}'|\right\} \end{aligned} \tag{2.9}$$

Finally, since $\mathscr{H}_0(\mathbf{P})$ and $\mathscr{N}(\mathbf{P})$ commute with each other, standard manipulation of Eq. (2.1) (see Appendix A) yields

$$\begin{aligned} &\langle\mathbf{k}|T_S(\mathbf{P}, \omega + i\eta)|\mathbf{k}'\rangle - \langle\mathbf{k}|T_S(\mathbf{P}, \omega - i\eta)|\mathbf{k}'\rangle \\ &\quad = \int \frac{d^3q}{(2\pi)^3}\langle\mathbf{k}|T_S(\mathbf{P}, \omega \pm i\eta)|\mathbf{q}\rangle N(\mathbf{P}, \mathbf{q}) \\ &\quad \times\left[\frac{1}{\omega + i\eta - \Omega(\mathbf{P}, \mathbf{q})} - \frac{1}{\omega - i\eta - \Omega(\mathbf{P}, \mathbf{q})}\right]\langle\mathbf{q}|T_S(\mathbf{P}, \omega \mp i\eta)|\mathbf{k}'\rangle \end{aligned} \tag{2.10}$$

so that Eq. (2.9) becomes

$$\begin{aligned} \langle\mathbf{k}|\Gamma_S(\mathbf{P}, \omega)|\mathbf{k}'\rangle &= \sum_\nu \langle\mathbf{k}|F_\nu(\mathbf{P}, S)|\mathbf{k}'\rangle\delta[\omega - E_\nu(\mathbf{P}, S)] \\ &\quad + \int \frac{d^3q}{(2\pi)^3}\langle\mathbf{k}|T_S(\mathbf{P}, \omega \pm i\eta)|\mathbf{q}\rangle N(\mathbf{P}, \mathbf{q}) \\ &\quad \times\delta[\omega - \Omega(\mathbf{P}, \mathbf{q})]\langle\mathbf{q}|T_S(\mathbf{P}, \omega \mp i\eta)|\mathbf{k}'\rangle \end{aligned} \tag{2.11}$$

[In Eqs. (2.10) and (2.11) either the upper or lower choice of sign of $i\eta$ is implied.]

Equation (2.11) relates $\Gamma_S(\mathbf{P}, \omega)$, and with an appropriate choice of z', $T_S(\mathbf{P}, z)$ through Eq. (2.8), entirely to on-energy-shell quantities. For subsequent analysis it is often necessary to perform discretized Fourier sums over the variable z.(6) The spectral representation of $T_S(\mathbf{P}, z)$, Eq. (2.8), facilitates these operations and puts the physical information into the spectral function $\Gamma_S(\mathbf{P}, \omega)$.

The on-shell matrix elements of T_S *define* internal composite wave functions in the medium. In analogy with standard definitions for the medium independent case, a scattering amplitude $f_{\mathbf{P}S}(\mathbf{k},\mathbf{k}')$ can be defined by

$$f_{\mathbf{P}S}(\mathbf{k},\mathbf{k}') \equiv \langle\mathbf{k}|T_S(\mathbf{P},\Omega(\mathbf{P},\mathbf{k}')+i\eta)|\mathbf{k}'\rangle \tag{2.12}$$

This amplitude $f_{\mathbf{P}S}$ satisfies several relations familiar from standard medium-independent scattering theory. These relations follow from the general properties of T (see Appendix A). The corresponding wave function

$$\Phi_{\mathbf{P}S\mathbf{k}'}(\mathbf{k}) \equiv (2\pi)^3\delta(\mathbf{k}-\mathbf{k}') + \frac{N(\mathbf{P},\mathbf{k})}{\Omega(\mathbf{P},\mathbf{k}')-\Omega(\mathbf{P},\mathbf{k})+i\eta}\, f_{\mathbf{P}S}(\mathbf{k},\mathbf{k}') \tag{2.13}$$

then satisfies the Schrödinger equation

$$\begin{aligned}\Phi_{\mathbf{P}S\mathbf{k}'}(\mathbf{k}) &= (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\\ &\quad+\frac{N(\mathbf{P},\mathbf{k})}{\Omega(\mathbf{P},\mathbf{k}')-\Omega(\mathbf{P},\mathbf{k})+i\eta}\int\frac{d^3q}{(2\pi)^3}\langle\mathbf{k}|\mathbf{V}_S|\mathbf{q}\rangle\Phi_{\mathbf{P}S\mathbf{k}'}(\mathbf{q})\end{aligned} \tag{2.14}$$

Turning to the bound state wave functions, we first note that Eqs. (2.1) and (2.7) require the pole contributions to the T matrix to satisfy the homogeneous equation

$$\langle\mathbf{k}|F_\nu(\mathbf{P},S)|\mathbf{k}'\rangle = \int\frac{d^3q}{(2\pi)^3}\langle\mathbf{k}|V_S|\mathbf{q}\rangle\frac{N(\mathbf{P},\mathbf{q})}{E_\nu-\Omega(\mathbf{P},\mathbf{q})}\langle\mathbf{q}|F_\nu(\mathbf{P},S)|\mathbf{k}'\rangle \tag{2.15}$$

with the normalization condition

$$\langle\mathbf{k}|F_\nu(\mathbf{P},S)|\mathbf{k}'\rangle = \int\frac{d^3q}{(2\pi)^3}\langle\mathbf{k}|F_\nu(\mathbf{P},S)|\mathbf{q}\rangle\frac{N(\mathbf{P},\mathbf{q})}{[E_\nu-\Omega(\mathbf{P},\mathbf{q})]^2}\langle\mathbf{q}|F_\nu(\mathbf{P},S)|\mathbf{k}'\rangle \tag{2.18}$$

Furthermore, the matrix elements of F_ν have the same symmetry as those of T_S, i.e., $\langle\mathbf{k}|F_\nu(\mathbf{P},S)|\mathbf{k}'\rangle = \langle\mathbf{k}'|F_\nu(\mathbf{P},S)|\mathbf{k}\rangle$. We then define bound state wave functions $\phi_{\mathbf{P}S\nu}(\mathbf{k})$ by

$$\langle\mathbf{k}|F_\nu(\mathbf{P},S)|\mathbf{k}'\rangle = \frac{E_\nu-\Omega(\mathbf{P},\mathbf{k})}{N(\mathbf{P},\mathbf{k})}\,\frac{\Phi_{\mathbf{P}S\nu}(\mathbf{k})\Phi^*_{\mathbf{P}S\nu}(\mathbf{k}')}{\eta_\nu}\,\frac{E_\nu-\Omega(\mathbf{P},\mathbf{k}')}{N(\mathbf{P},\mathbf{k}')} \tag{2.17}$$

and the bound state Schrödinger equation is

$$\Phi_{\mathbf{P}S\nu}(\mathbf{k}) = \frac{N(\mathbf{P},\mathbf{k})}{E_\nu(\mathbf{P},S)-\Omega(\mathbf{P},\mathbf{k})}\int\frac{d^3q}{(2\pi)^3}\langle\mathbf{k}|V_S|\mathbf{q}\rangle\Phi_{\mathbf{P}S\nu}(\mathbf{q}) \tag{2.18}$$

These medium-inclusive wave functions satisfy modified orthonormality and completeness conditions, to wit

$$\begin{aligned}
&\int \frac{d^3k}{(2\pi)^3}\, \Phi^*_{\mathbf{P}S\nu}(\mathbf{k})\, \frac{1}{N(\mathbf{P},\mathbf{k})}\, \Phi_{\mathbf{P}S\nu'}(\mathbf{k}) = \eta_\nu \delta_{\nu\nu'} \\
&\int \frac{d^3k}{(2\pi)^3}\, \Phi^*_{\mathbf{P}S\nu}(\mathbf{k})\, \frac{1}{N(\mathbf{P},\mathbf{k})}\, \Phi_{\mathbf{P}S\mathbf{q}}(\mathbf{k}) = 0 \qquad (2.19) \\
&\int \frac{d^3k}{(2\pi)^3}\, \Phi^*_{\mathbf{P}S\mathbf{q}}(\mathbf{k})\, \frac{1}{N(\mathbf{P},\mathbf{k})}\, \Phi_{\mathbf{P}S\mathbf{q}'}(\mathbf{k}) = (2\pi)^3 \frac{\delta(\mathbf{q}-\mathbf{q}')}{N(\mathbf{P},\mathbf{q})}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_\nu \frac{1}{\eta_\nu}\, \Phi_{\mathbf{P}S\nu}(\mathbf{k})\Phi^*_{\mathbf{P}S\nu}(\mathbf{k}') + \int \frac{d^3q}{(2\pi)^3}\, \Phi_{\mathbf{P}S\mathbf{q}}(\mathbf{k}) N(\mathbf{P},\mathbf{q}) \Phi^*_{\mathbf{P}S\mathbf{q}}(\mathbf{k}') \\
&\quad = (2\pi)^3 N(\mathbf{P},\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') \qquad (2.20)
\end{aligned}$$

The normalization constant η_ν, which appears explicitly in (2.17) and (2.20), is defined by Eq. (2.19) ($\nu = \nu'$). The internal composite states in the medium are then completely specified aside from this bound state normalization constant. We may then turn to the question of their relationship to the medium single-particle properties $n(\mathbf{k}_1)$ and $\epsilon(\mathbf{k}_1)$. It will prove useful, however, to have the T-matrix spectral function $\Gamma(\omega)$ given explicitly in terms of the bound state wave functions and the scattering amplitudes. Equation (2.11) can be rewritten, using Eqs. (2.12) and (2.17), as

$$\begin{aligned}
\langle \mathbf{k}|\Gamma_s(\mathbf{P},\omega)|\mathbf{k}'\rangle = {} & \sum_\nu \delta[\omega - E_\nu(\mathbf{P},S)]\, \frac{E_\nu(\mathbf{P},S) - \Omega(\mathbf{P},\mathbf{k})}{N(\mathbf{P},\mathbf{k})} \\
& \times \frac{E_\nu(\mathbf{P},S) - \Omega(\mathbf{P},\mathbf{k}')}{N(\mathbf{P},\mathbf{k}')}\, \frac{\Phi_{\mathbf{P}S\nu}(\mathbf{k})\Phi^*_{\mathbf{P}S\nu}(\mathbf{k}')}{\eta_\nu} \\
& + \int \frac{d^3q}{(2\pi)^3}\, \delta[\omega - \Omega(\mathbf{P},\mathbf{q})] N(\mathbf{P},\mathbf{q}) \\
& \times \frac{1}{2}\left\{ f_{\mathbf{P}S}(\mathbf{k},\mathbf{q}) f^*_{\mathbf{P}S}(\mathbf{k}',\mathbf{q}) + f^*_{\mathbf{P}S}(\mathbf{k},\mathbf{q}) f_{\mathbf{P}S}(\mathbf{k}',\mathbf{q}) \right\} \qquad (2.21)
\end{aligned}$$

If, in Eq. (2.8), we set $z' = \Omega(\mathbf{k}') \pm i\eta$, we have

$$\begin{aligned}
\langle \mathbf{k}|T_S(\mathbf{P},z)|\mathbf{k}'\rangle = {} & \frac{1}{2}\left[f_{\mathbf{P}S}(\mathbf{k},\mathbf{k}') + f^*_{\mathbf{P}S}(\mathbf{k},\mathbf{k}') \right] \\
& + \int_{-\infty}^{\infty} d\omega\, \langle \mathbf{k}|\Gamma_S(\mathbf{P},\omega)|\mathbf{k}'\rangle \left[\frac{1}{z-\omega} - \text{P.V.}\,\frac{1}{\Omega(\mathbf{P},\mathbf{k}') - \omega} \right] \\
& \qquad (2.22)
\end{aligned}$$

where P.V. denotes the principal value. Thus, the medium-dependent T matrix itself is given explicitly in terms of the bound state wave functions and scattering amplitude, in precise analogy with traditional medium-independent scattering theory.

Equations (2.1) and (2.12) give an equation [analogous to Eq. (2.14)] for the amplitude $f_{\mathbf{P}S}(\mathbf{k},\mathbf{k}')$. A similar equation can be written for the medium-independent amplitude $f^{(0)}(\mathbf{k},\mathbf{k}')$ (note that this is independent of $\mathbf{P}$) by taking $N(\mathbf{P},\mathbf{k})\rightarrow 1$ and $\Omega(\mathbf{P},\mathbf{k})\rightarrow k^2/m$. These two equations may then be combined in the result

$$\begin{aligned} f_{\mathbf{P}S}(\mathbf{k},\mathbf{k}') = f_S^0(\mathbf{k},\mathbf{k}') &+ \int \frac{d^3q}{(2\pi)^3} f_S^0(\mathbf{k},\mathbf{q}) \\ &\times \left[\frac{N(\mathbf{P},\mathbf{q})}{\Omega(\mathbf{P},\mathbf{k}') - \Omega(\mathbf{P},\mathbf{q}) + i\delta} - \frac{1}{k'^2/m - q^2/m + i\delta} \right] \\ &\times f_{\mathbf{P},S}(\mathbf{q},\mathbf{k}) \end{aligned} \tag{2.33}$$

The finite-temperature version of the familiar Galitskii equation$^{(8)}$ is obtained with the replacement of $\Omega(\mathbf{P},\mathbf{k})$ by its free particle value. Similarly $N(\mathbf{P},\mathbf{k})$ is also replaced by its free particle value, i.e., in terms of free particle Fermi distributions (or the theta function at zero temperature). It is to be anticipated that in further development of the medium-dependent theory, familiar results will be obtained by these replacements.

3. SINGLE-PARTICLE PROPERTIES AND COUPLED EQUATIONS

In relating the single-particle properties to the T matrix discussed in Section 2, we shall be guided by the traditional theory of finite-temperature many-body systems. Such a theory determines the single-particle momentum distribution $n(\mathbf{k}_1)$ and the single-particle energy $\epsilon(\mathbf{k}_1)$ to be

$$n(\mathbf{k}_1) = -iG_1(\mathbf{k}_1, t - t^+) \tag{3.1}$$

and

$$\epsilon(\mathbf{k}_1) = \frac{k_1^2}{2m} + \operatorname{Re}\Sigma(\mathbf{k}_1, \epsilon(\mathbf{k}_1) \pm i\delta) \tag{3.2}$$

G_1 is the single-particle propagator (in the time domain with thermal boundary conditions). Σ is the single-particle self-energy, given by the Dyson equation

$$\Sigma = G_{10}^{-1} - G_1^{-1} \tag{3.3}$$

with G_{10} the free single-particle propagator (G_{10} and G_1 here in the energy domain). Σ and G_1 are further related in a T-matrix theory by

$$\Sigma = TG_1 \tag{3.4}$$

where T represents a T matrix, generally of the form

$$T = V + V\Lambda T \tag{3.5}$$

Here, the choice of Λ, the pair propagation for intermediate states, has various possibilities. The choice $\Lambda = G_1 G_1$, with fully dressed single-particle propagators G_1, leads to a formally conserving approximation in the sense discussed by Baym and Kadanoff,(15) although different choices have been employed.(16) Each of these choices generally results in off-energy-shell denominators and therefore a structure more complicated than Eq. (2.1). We wish to retain Eq. (2.1), however, since it emerges most naturally from the composite formulation. Consequently, we ask for the sequence of approximations which produces Eq. (2.1) from Eq. (3.5). Taking $\Lambda = G_1 G_1$ and using zeroth-order perturbation theory, one obtains a structure like Eq. (2.1) except that $\mathscr{H}_0$ becomes an entirely free pair Hamiltonian, while $\mathscr{N}$ becomes the blocking factor appropriate to free Fermions. Alternatively, replacement of off-energy-shell denominators by their on-shell values [Eq. (3.2)] gives the correct energy denominator in Eq. (2.1) if we assume the energies in Eq. (2.4) to be identified with the sum of single-particle energies, Eq. (3.2). This same replacement gives the correct $N(\mathbf{P})$ factor, Eq. (2.5), in the numerator if we use the equivalent of Eq. (3.1) in the energy domain, along with the normalization requirement $G_1(\mathbf{k}, 0+) - G_1(\mathbf{k}, 0-) = -i$, which results from the commutation relations. The key is that at some point in the traditional analysis, one must replace off-shell by on-shell energy denominators.

In Eq. (3.4) we now identify T with the composite form, Eq. (2.1), using the structure exhibited in Eq. (2.8). Equations (3.4), (3.1), and (3.2) together then give the single-particle energy if we are guided in making further on-shell replacement by the notion that our result should relate $\epsilon(\mathbf{k})$ directly to the physical $n(\mathbf{k})$ and the pair amplitudes and bound states. The result is

$$\begin{aligned}\epsilon(\mathbf{k}_1) = \frac{k_1^2}{2m} + \int \frac{d^3k_2}{(2\pi)^3} \sum_S D_S \\ \times \Big\{ n(\mathbf{k}_2)\mathrm{Re}\big[f_{\mathbf{P}S}(\mathbf{k},\mathbf{k}) + (-1)^S f_{\mathbf{P}S}(-\mathbf{k},\mathbf{k}) \big] \\ + \mathrm{P.V.}\int d\omega\, f_B(\omega)\, \frac{1}{\Omega(\mathbf{P},\mathbf{k}) - \omega} \\ + \big[\langle \mathbf{k}|\Gamma_S(\mathbf{P},\omega)|\mathbf{k}\rangle + (-1)^S \langle -\mathbf{k}|\Gamma_S(\mathbf{P},\omega)|\mathbf{k}\rangle \big] \Big\}\end{aligned} \tag{3.6}$$

where $f_B(\omega)$ is the Bose function

$$f_B(\omega) = \left[e^{\beta(\omega - 2\mu)} - 1\right]^{-1}$$

and μ is the chemical potential. The spin factor D_S is

$$D_S = \tfrac{1}{2}\delta_{S,0} + \tfrac{3}{2}\delta_{S,1}$$

Equations (3.1) and (3.3) together then give, in a similar manner, the single-particle momentum distribution,

$$\begin{aligned} n(\mathbf{k}_1) = f_F(\epsilon(\mathbf{k}_1)) &+ \int \frac{d^3k_2}{(2\pi)^3} \sum_S D_S \\ &\times \text{P.V.} \int d\omega \frac{\left[f_B(\omega) - f_B(\Omega(\mathbf{P},\mathbf{k}))\right]}{\left[\Omega(\mathbf{P},\mathbf{k}) - \omega\right]^2} \\ &\times \left[\langle \mathbf{k}|\Gamma_S(\mathbf{P},\omega)|\mathbf{k}\rangle + (-1)^S \langle -\mathbf{k}|\Gamma_S(\mathbf{P},\omega)|\mathbf{k}\rangle\right] N(\mathbf{P},\mathbf{k}) \end{aligned} \tag{3.7}$$

with $f_F(\epsilon)$ the Fermi function

$$f_F(\epsilon) = \left[e^{\beta(\epsilon - \mu)} + 1\right]^{-1}$$

Equations (3.6) and (3.7) form the fundamental connection between the wave functions (both bound and scattering states) in the medium and the single-particle properties. These wave functions indeed arose from consideration of the medium as a system of composites. The effective two-body Schrödinger equation in the medium followed from the T matrix (not the reverse) and the T matrix itself followed from a Hartree–Fock-like approximation on a higher-order composite propagator. The composite formulation in itself, however, proved unable to give the single-particle properties, since the true single-particle nature of the medium had been suppressed.

In accord with the program proposed in Section 1, we must then verify that Eqs. (3.6) and (3.7) do in fact recover the usual perturbation expansion to second order in the potential for all densities and the fugacity expansion to second order in the density for all potential strengths. Finally, the results should afford some meaningful physical interpretation. We examine first the perturbation expansion.

In the perturbation expansion there is no bound state contribution. Furthermore, the scattering amplitude leading term is first order in the potential, while the leading term in the spectral function Γ_S is second order. Hence, in Eqs. (3.6) and (3.7), in all terms containing Γ_s the energy denominators may be replaced by their noninteracting values, i.e.,

$$\Omega(\mathbf{P},\mathbf{k}) \to \frac{P^2}{4m} + \frac{k^2}{m}$$

Furthermore, Eq. (3.7) shows that $n(\mathbf{k}_1) = f_F(\epsilon_1)$ to first order in the coupling constant, so that in all terms we may use the replacement

$$N(P,k) \to 1 - f_F\left[\frac{1}{2m}\left(\frac{1}{2}\mathbf{P} + \mathbf{k}\right)^2\right] - f_F\left[\frac{1}{2m}\left(\frac{1}{2}\mathbf{P} - \mathbf{k}\right)^2\right]$$

These replacements are also used in the spectral function, Eq. (2.21), where the scattering amplitude is simply replaced by the potential. With these substitutions, Eqs. (3.6) and (3.7), when expanded to second order, give correctly the perturbative expressions(17) for $\epsilon(\mathbf{k}_1)$ and $n(\mathbf{k}_1)$ (see Appendix B).

Before consideration of the fugacity expansion, we first rewrite Eq. (3.7). Note that the "noninteracting" state $|\mathbf{k}\rangle$ is understood to be characterized by an internal momentum $\mathbf{k}$ as well as total momentum $\mathbf{P}$ and total spin S, and in fact is a plane wave eigenstate of the Hamiltonian $\mathscr{H}_0(\mathbf{P})$ in Eq. (2.2) with eigenvalue $\Omega(\mathbf{P},\mathbf{k})$,

$$\mathscr{H}_0(\mathbf{P})|\mathbf{k}\rangle = \Omega(\mathbf{P},\mathbf{k})|\mathbf{k}\rangle \tag{3.8}$$

The full Hamiltonian $\mathscr{H}$ is that for which the states $|\nu\rangle$ and $|\mathbf{q}\rangle$ are eigenstates with energies $E_\nu(\mathbf{P},S)$ and $\Omega(\mathbf{P},\mathbf{q})$, respectively,

$$\mathscr{H}|\nu\rangle = E_\nu(\mathbf{P},S)|\nu\rangle, \qquad \mathscr{H}|\mathbf{q}\rangle = \Omega(\mathbf{P},\mathbf{q})|\mathbf{q}\rangle \tag{3.9}$$

The wave functions $\phi_{\mathbf{P}S\nu}(\mathbf{k})$ and $\phi_{\mathbf{P}S\mathbf{q}}(\mathbf{k})$ discussed in Section 2 are

$$\langle\mathbf{k}\,|\,\nu\rangle = \phi_{\mathbf{P}S\nu}(\mathbf{k}), \qquad \langle\mathbf{k}\,|\,\mathbf{q}\rangle = \phi_{\mathbf{P}S\mathbf{q}}(\mathbf{k}) \tag{3.10}$$

With the aid of Eqs. (2.13), (2.20), (2.21), (3.8), (3.9), and (3.10), it is possible to show that [again $f_B(E)$ is the Bose function]

$$\begin{aligned}
&\langle\mathbf{k}|f_B(\mathscr{H}) - f_B(\mathscr{H}_0)|\mathbf{k}'\rangle \\
&\quad = \frac{1}{2}\left[N(\mathbf{P},\mathbf{k}')f_{\mathbf{P}S}(\mathbf{k},\mathbf{k}') + N(\mathbf{P},\mathbf{k})f^*_{\mathbf{P}S}(\mathbf{k}',\mathbf{k})\right] \\
&\quad\quad \times \frac{f_B[\Omega(\mathbf{P},\mathbf{k}')] - f_B[\Omega(\mathbf{P},\mathbf{k})]}{\Omega(\mathbf{P},\mathbf{k}') - \Omega(\mathbf{P},\mathbf{k})} + \frac{1}{2}\int d\omega\,\langle\mathbf{k}|\Gamma_S(\mathbf{P},\omega)|\mathbf{k}'\rangle \\
&\quad\quad \times \left\{\frac{f_B(\omega) - f_B[\Omega(\mathbf{P},\mathbf{k})]}{\omega - \Omega(\mathbf{P},\mathbf{k})}\,\frac{1}{\omega - \Omega(\mathbf{P},\mathbf{k}) - i\delta}\right. \\
&\quad\quad \left. + \frac{f_B(\omega) - f_B[\Omega(\mathbf{P},\mathbf{k}')]}{\omega - \Omega(\mathbf{P},\mathbf{k}')}\,\frac{1}{\omega - \Omega(\mathbf{P},\mathbf{k}') + i\delta}\right\}
\end{aligned} \tag{3.11}$$

From the diagonal matrix element of Eq. (3.11), we recognize that Eq. (3.7) can be written in the form

$$n(\mathbf{k}_1) = f_F(\epsilon(\mathbf{k}_1)) + \int \frac{d^3k_2}{(2\pi)^3} \sum_S D_S$$
$$\times \left\{ -N(\mathbf{P},\mathbf{k}) \frac{d}{dE} f_B(E) \Big|_{E=\Omega(\mathbf{P},\mathbf{k})} \mathrm{Re}\left[f_{\mathbf{P}S}(\mathbf{k},\mathbf{k}) + (-1)^S f_{\mathbf{P}S}(-\mathbf{k},\mathbf{k}) \right] \right.$$
$$\left. + \langle \mathbf{k} + (-1)^S(-\mathbf{k}) | f_B(\mathscr{H}) - f_B(\mathscr{H}_0) | \mathbf{k} \rangle \right\} \tag{3.12}$$

The fugacity expansion parameter is $\zeta = e^{\beta\mu}$, with small ζ associated with low density and/or high temperature. In expanding Eq. (3.12) to second order in ζ, it must be noted that the Fermi function $f_F(\epsilon(\mathbf{k}_1))$ requires also that $\epsilon(\mathbf{k}_1)$ be expanded in ζ as well through Eq. (3.6). The end result then gives the fugacity expansion for $n(\mathbf{k}_1)$,

$$n(\mathbf{k}_1) = \zeta e^{-\beta k_1^2/2m}$$
$$+ \zeta^2 \left\{ -e^{-\beta k_1^2/m} \right.$$
$$\left. + \int \frac{d^3k_2}{(2\pi)^3} \sum_S D_S \langle \mathbf{k} + (-1)^S(-\mathbf{k}) | e^{-\beta\mathscr{H}} - e^{-\beta\mathscr{H}_0} | \mathbf{k} \rangle \right\} \tag{3.13}$$

In the second term, $\mathscr{H}$ and $\mathscr{H}_0$ are taken at the $\zeta = 0$ medium-independent limit. The usual fugacity expansion gives the total density as a series in ζ. Integration of Eq. (3.13) [see Eq. (2.6)] then gives the usual result.(3)

Knowing that Eqs. (3.6) and (3.7) do indeed recover the required limiting cases, we then rewrite the density and energy as

$$n(\mathbf{k}_1) = f_F[\epsilon(\mathbf{k}_1)] + \int \frac{d^3k_2}{(2\pi)^3} \sum_S D_S$$
$$\times \left\{ \int \frac{d^3q}{(2\pi)^3} N(\mathbf{P},\mathbf{q}) \left[|f_{\mathbf{P}S}(\mathbf{k},\mathbf{q})|^2 + (-1)^S \mathrm{Re}\, f_{\mathbf{P}S}(-\mathbf{k},\mathbf{q}) f^*_{\mathbf{P}S}(\mathbf{k},\mathbf{q}) \right] \right.$$
$$\times N(\mathbf{P},\mathbf{k}) \frac{f_B[\Omega(\mathbf{P},\mathbf{q})] - f_B[\Omega(\mathbf{P},\mathbf{k})]}{[\Omega(\mathbf{P},\mathbf{q}) - \Omega(\mathbf{P},\mathbf{k})]^2}$$
$$+ \sum_\nu \frac{|\phi_{\mathbf{P}S\nu}(\mathbf{k})|^2 + (-1)^S \mathrm{Re}\, \phi^*_{\mathbf{P}S\nu}(-\mathbf{k}) \phi_{\mathbf{P}S\nu}(\mathbf{k})}{N(\mathbf{P},\mathbf{k})\eta_\nu}$$
$$\left. \times \left[f_B[E_\nu(\mathbf{P},S)] - f_B[\Omega(\mathbf{P},\mathbf{k})] \right] \right\} \tag{3.14}$$

and

$$
\begin{aligned}
\epsilon(\mathbf{k}_1) = \frac{k_1^2}{2m} + \int \frac{d^3k_2}{(2\pi)^3} \sum_S D_S \\
\times \Bigg\{ n(\mathbf{k}_2)\mathrm{Re}\left[f_{\mathbf{P}S}(\mathbf{k},\mathbf{k}) + (-1)^S f_{\mathbf{P}S}(-\mathbf{k},\mathbf{k}) \right] \\
+ \mathrm{P.V.}\int \frac{d^3q}{(2\pi)^3} \left[|f_{\mathbf{P}S}(\mathbf{k},\mathbf{q})|^2 + (-1)^S \mathrm{Re}\, f_{\mathbf{P}S}(-\mathbf{k},\mathbf{q}) f^*_{\mathbf{P}S}(\mathbf{k},\mathbf{q}) \right] \\
\times N(\mathbf{P},\mathbf{q}) \frac{f_B[\Omega(\mathbf{P},\mathbf{q})]}{\Omega(\mathbf{P},\mathbf{k}) - \Omega(\mathbf{P},\mathbf{q})} + \sum_\nu \frac{\Omega(\mathbf{P},\mathbf{k}) - E_\nu(\mathbf{P},S)}{N(\mathbf{P},\mathbf{k})\eta_\nu} \\
\times \left[|\phi_{\mathbf{P}S\nu}(\mathbf{k})|^2 + (-1)^S \mathrm{Re}\,\phi^*_{\mathbf{P}S\nu}(-\mathbf{k})\phi_{\mathbf{P}S\nu}(\mathbf{k}) \right] f_B[E_\nu(\mathbf{P},S)] \Bigg\}
\end{aligned}
\tag{3.15}
$$

The four coupled equations (2.18), (2.23), (3.14), and (3.15), together with the density $n(\beta, \mu)$ given by

$$n(\beta, \mu) = 2\int \frac{d^3k_1}{(2\pi)^3} n(\mathbf{k}_1)$$

form a complete description of the medium-dependent dynamics and the many-body thermodynamics, deduced from the view of the many-Fermion medium as a set of pair-composites.

We may apply this theory to the special case of a repulsive Fermi gas, studied previously by Galitskii for zero temperature.(8) In this case there are no bound states. Galitskii's result gives the chemical potential in terms of the density n, $n = k_f^3/3\pi^2$, through the expression $\mu = \epsilon(k_f)$. We may recapture this result from our formulation through the following steps in Eq. (3.25). (1) Replace the scattering amplitude $f_{\mathbf{P}S}(\mathbf{k},\mathbf{q})$ by the medium-independent amplitude throughout. (2) Replace the energy denominator by its free value, i.e., $\Omega(\mathbf{P},\mathbf{k}) - \Omega(\mathbf{P},\mathbf{q}) \to k^2/m - q^2/m$. (3) Replace

$$n(\mathbf{k}_2) \to f_F[\epsilon(\mathbf{k}_2)] \to \Theta(k_f - k_2)$$

and

$$
\begin{aligned}
N(\mathbf{P},\mathbf{q}) f_B[\Omega(\mathbf{P},\mathbf{q})] &\to f_F(\epsilon(\mathbf{P}/2 + \mathbf{q})) f_F(\epsilon(\mathbf{P}/2 - \mathbf{q})) \\
&\to \Theta[k_f - |\mathbf{P}/2 + \mathbf{q}|]\Theta[k_f - |\mathbf{P}/2 - \mathbf{q}|]
\end{aligned}
$$

(see Appendix B). (4) In this low-density limit only the zero-energy singlet scattering amplitude enters, and this is simply the scattering length a,

$$a = \frac{m}{4\pi\hbar^2} \lim_{|k|\to 0} f^{(0)}_{S=0}(\mathbf{k}, \mathbf{k}')|_{|\mathbf{k}'|=|\mathbf{k}|}$$

Note that in this formulation the medium dependence of the two-body dynamics is not at all required to recover the Galitskii result.

4. CONCLUSIONS AND INTERPRETATIONS

Equations (2.18), (2.23), (3.14), and (3.15) represent a mixed description of the many-body dynamics and its attendant thermodynamic properties. This description uses both the single-particle properties of the medium, and its composite nature as well. The quantities $n(\mathbf{k}_1)$ and $\epsilon(\mathbf{k}_1)$, as they emerge from the composite formulation, required interpretation beyond the composite description. This interpretation was provided from the more traditional many-body theory, in terms of single-particle propagation. Clearly for an understanding of the two-body dynamics in the medium one must also understand the single-particle properties, in particular the momentum distribution $n(\mathbf{k}_1)$ and the energy $\epsilon(\mathbf{k}_1)$. These quantities enter through the blocking operator $\mathcal{N}$ and the "noninteracting" Hamiltonian $\mathcal{H}_0$ in Section 2, and their appearance in the dynamical equations accords with one's physical intuition. But the converse is also true. An understanding of the single-particle properties requires an understanding of the composite dynamics, and this is seen in Section 3. In this case, however, the structure of Eqs. (3.14) and (3.15) are not such as to provide a simple physical picture of their dynamical effect. In fact, we do not need the details of the single-particle motion to obtain thermodynamic quantities, since the full particle density $n(\beta, \mu)$ is given entirely in terms of the blocking operator,

$$n(\beta, \mu) = \int \frac{d^3k}{(2\pi)^3} \{1 - N(\mathbf{P}, \mathbf{k})\}$$

for all $\mathbf{P}$. That is, to obtain $n(\beta, \mu)$, one need only consider composites at rest, and sum over their internal momenta.

Nonetheless, if we consider Eq. (3.12) it is possible to picture corrections to free quasiparticle behavior due to composite formation. Equation (3.12) contains, in addition to the Fermi function for the free quasiparticles, two additional terms. The second of these exhibits the difference of the Bose function for the interacting and free but fully dressed quasiparticles, $f_B(\mathcal{H}) - f_B(\mathcal{H}_0)$. This corresponds to the two-particle density matrix corrections to the familiar quantum cluster expansion$^{(3)}$ with, however, the following differences. In the usual cluster expansion, the Bose functions are replaced by Boltzmann factors, and medium dependence of $\mathcal{H}$ and $\mathcal{H}_0$

does not appear. The fact that Bose functions appear here reflects the fact that the composites *are* Bosons (not elementary Bosons, since there are additional symmetries). In fact, the cluster expansion never introduces the concept of the single quasiparticle. Consequently, the only remaining term in the fugacity expansion consists of the second-order terms from the free particle Fermi function.

The last term of Eq. (3.12) is in fact the two-quasiparticle density matrix for a quantum quasiparticle cluster expansion. The zeroth term $f(\epsilon(\mathbf{k}_1))$ here represents the free quasiparticle motion, with an energy which already includes to some extent the interactions. Since the last term explicitly gives the two-quasiparticle cluster correction, this interaction has been overcounted, and the second term of Eq. (3.12) then corrects this overcounting. In fact, in the fugacity expansion, the first two terms of Eq. (3.12) simply replace the quasiparticles by undressed (free) particles, independent of the interaction strength. A procedure similar to that giving $n(\mathbf{k}_1)$ in Eq. (3.12) leads to an expression for $\epsilon(\mathbf{k}_1)$ alternative to Eq. (3.6), namely,

$$\begin{aligned}\epsilon(\mathbf{k}_1) = \frac{k_1^2}{2m} + \int \frac{d^3k_2}{(2\pi)^3} \sum_S D_S \\ \times \Big\{ \big[n(\mathbf{k}_2) + f_B[\Omega(\mathbf{P},\mathbf{k})] \big] \mathrm{Re}\big[f_{\mathbf{P}S}(\mathbf{k},\mathbf{k}) + (-1)^S f_{\mathbf{P}S}(-\mathbf{k},\mathbf{k}) \big] \\ - \frac{1}{2} \langle \mathbf{k} + (-1)^S(-\mathbf{k}) | f_B(\mathscr{H})(\mathscr{H} - \mathscr{H}_0) \frac{1}{\mathscr{N}(\mathbf{P})} \\ + \frac{1}{\mathscr{N}(\mathbf{P})} (\mathscr{H} - \mathscr{H}_0) f_B(\mathscr{H}) | \mathbf{k} \rangle \Big\} \qquad (4.1)\end{aligned}$$

$\mathscr{N}(\mathbf{P})$ is the blocking operator defined in Section 2. Note that $\mathscr{H} - \mathscr{H}_0 = \mathscr{N} V$, and the last term gives the two-quasiparticle cluster expansion for the quasiparticle energy. The first correction, involving $n(\mathbf{k}_2)$, is recognized as the usual Hartree–Fock-like self-energy. Here again the medium dependence has been overcounted in constructing the cluster terms, and the term involving $f_B[\Omega(\mathbf{P},\mathbf{k})]$ corrects for this overcounting.

The cluster terms in Eqs. (3.12) and (4.1) have a clear physical interpretation, while those terms reflecting the overcounting are not at all transparent. In fact, from a practical point of view, the set of coupled equations (2.18), (2.23), (3.14), and (3.15) [together with (2.4) and (2.5)] are more useful, while Eqs. (3.12) and (4.1) do afford a physical interpretation.

As was mentioned, a possible application for this theoretical treatment lies in the two-dimensional He^3 gas problem. If one takes the density to be sufficiently low that the blocking factors $N(\mathbf{P},\mathbf{k})$ are taken as unity, the theory simplifies greatly. In particular, one can show that all quantities can be expressed in terms of the forward medium-independent scattering amplitudes $f^{(0)}(\pm\mathbf{k},\mathbf{k})$, and hence in terms of the phase shifts of the two-body potential and the bound state energies. In this respect the theory is as simple as the fugacity expansion. However, a detailed investigation of the He^3 two-dimensional system with these simplifications reveals thermodynamic inconsistencies at low temperature.(18) The difficulties can be traced to a Bose-like condensation which in fact results from the blocking factors being replaced by 1. That is, medium dependence plays an essential role over at least some region in the temperature–density plane. The full medium-dependent theory has not yet been analyzed for this system, although it appears that such analysis is necessary for complete consistency in the two-dimensional problem. It would appear that if one considers the entire range of temperatures, for any interesting physical system strictly medium-independent dynamics never suffice.

Application of the theory to helium monolayer systems seems feasible, and because of the experimental accessibility of a relatively "low-density" gaseous phase for selected temperatures, we might expect the theory to have a wider range of applicability here than in the bulk case. Nevertheless, a second application, which will constitute a parallel effort, concerns bulk He^3 properties near the gas–liquid transition temperature. We wish to explore this for two reasons. First is the availability of considerable experimental data, and second is the fact that the direct calculation of properties like the gas–liquid transition temperature will severely test the theory in a case of practical interest.

Before attempting these calculations, however, it would appear to be useful to consider the simpler if somewhat artificial system of the hard sphere Fermi gas. In the near future, quantum mechanical molecular dynamics simulations of this system at finite temperature should provide an "exact" equation of state(19) for comparison with our analysis. Such a calculation for the hard sphere system, involving the self-consistent solution of the coupled equations (2.23), (3.6), and (3.7), is now in progress. It should be noted that this calculation itself provides a significant improvement over those in which the medium-dependent amplitudes involve either free particle energies or simple Fermi distributions (see Section 2). Furthermore, it will be interesting to compare higher-density corrections from the correlations included in our theory with those known corrections due to three-particle correlations.

This paper represents in a sense the completion of the conceptual framework begun in Ref. 7, where the self-consistent composite (Fermion pair) wave functions in the medium are constructed. This composite wave function [see Eqs. (2.13) and (2.18)] is analogous to the self-consistent constituent wave function of elementary Hartree–Fock theory. For example, there are no finite lifetime effects for the pair in our formulation, just as Hartree–Fock theory exhibits no lifetime effects for the single particle. Ordinary Hartree-Fock theory (in a translationally invariant system) requires the calculation of two basic quantities; one is the self-consistent single-particle energy and the other is the single-particle momentum distribution. Since the latter is trivial in this case, only one self-consistent equation need be solved. The full thermodynamic description is then provided by the momentum distribution. In the theory presented here, there are also two self-consistently determined quantities (neglecting the complication of bound states for purposes of this discussion). These are $\Omega(\mathbf{P},\mathbf{k})$ and $N(\mathbf{P},\mathbf{k})$, which are interpreted (section 3) via the single constituent energies and momentum distribution. This interpretation allows us to overcome a major stumbling block to a practical solution in Ref. 7. It is shown in Ref. 7 how the medium-dependent pair states depend on Ω and N. However, the specific dependence of Ω and N on these same pair states is not made explicit. In this paper, we have obtained coupled equations for Ω and N in terms of the amplitude $f_P(\mathbf{k},\mathbf{k}')$, and the self-consistent circle is made complete. Again, the full thermodynamic description is completed by utilizing the interpretation of N.

We would like to interpret this theory, therefore, as the complete equivalent, for Fermion pairs, of Hartree–Fock theory for single Fermions. Such a theory should be consistent with all exact limiting cases in which only pair correlations must be included, and we have seen that this is the case for three such limits; i.e., the low-density expansion at $T=0$, the coupling constant expansion for all T, and the small-fugacity (virial) expansion. The fact that we recover familiar results in certain limiting cases is not a major point. However, it lends a measure of confidence in solutions outside these known limits. Such solutions, particularly for the test case of the hard core gas, as discussed above, are in progress.

APPENDIX A

We show here some of the properties of the T matrix defined by Eq. (2.1). These properties are analogous to those of the medium-independent T matrix and are reproduced here to elucidate the specific medium dependence. While T depends on $\mathbf{P}$ and S, this dependence will be suppressed in the notation, $\mathbf{P}$ and S being taken as fixed parameters. Equation (2.1) is of the form

$$T(z) = V + VG(z)T(z) \tag{A.1}$$

with

$$G(z) = \mathscr{N}(z - \mathscr{H}_0)^{-1} \tag{A.2}$$

$\mathcal{N}$ and $\mathcal{H}_0$ are each Hermitian and commute with each other so that

$$G^+(z) = G(z^*) \tag{A.3}$$

Iteration of Eq. (A.1) to all orders and resummation gives an alternative equation for $T(z)$:

$$T(z) = V + T(z)G(z)V \tag{A.4}$$

But Hermitian conjugation of (A.1), together with (A.3), yields (A.4) for $T^+(z^*)$, so that

$$T^+(z) = T(z^*) \tag{A.5}$$

Comparison of Eq. (A.1) for $\langle \mathbf{k}|T(z)|\mathbf{k}'\rangle$ and (A.4) for $\langle \mathbf{k}'|T(z)|\mathbf{k}\rangle$ with the use of (A.5) and the symmetry of V, $\langle \mathbf{k}|V|\mathbf{k}'\rangle = \langle \mathbf{k}'|V|\mathbf{k}\rangle$, demonstrates the the symmetry of $T(z)$:

$$\langle \mathbf{k}|T(z)|\mathbf{k}'\rangle = \langle \mathbf{k}'|T(z)|\mathbf{k}\rangle \tag{A.6}$$

From Eq. (A.1),

$$T(z) - T(z^*) = V\{G(z) - G(z^*)\} + VG(z^*)\{T(z) - T(z^*)\}$$

and

$$T(z) - T(z^*) = \{1 - VG(z^*)\}^{-1}V\{G(z) - G(z^*)\}T(z)$$

Since $T(z^*) = \{1 - VG(z^*)\}^{-1}V$, we see that

$$T(z) - T(z^*) = T(z^*)\{G(z) - G(z^*)\}T(z) \tag{A.7}$$

Upon setting $z = \omega \pm i\delta$, Eq. (2.10) follows.

Equations (2.15) result from the analytic form of $T(z)$ shown in Eq. (2.7),

$$T(z) = \tilde{T}(z) + \sum_\nu \frac{F_\nu}{z - E_\nu} \tag{A.8}$$

where $T(z)$ is analytic in the vicinity of the poles at E_ν. With this analyticity ansatz, contour integration of (A.1) and (4) lead to

$$F_\nu = VG(E_\nu)F_\nu = F_\nu G(E_\nu)V \tag{A.9}$$

i.e., Eq. (2.15). The homogeneity of this equation implies that F can be chosen to be real. Finally, contour integration of (A.7), using this form for $T(z)$, gives Eq. (2.16),

$$F_\nu = F_\nu \frac{\mathcal{N}}{(E_\nu - \mathcal{H}_0)^2} F_\nu \tag{A.10}$$

The scattering amplitude [Eq. (2.12)] is

$$f(\mathbf{k},\mathbf{k}') \equiv \langle \mathbf{k}|T(\Omega(\mathbf{k}') + i\eta)|\mathbf{k}'\rangle \tag{A.11}$$

so that the energy corresponds to the right-hand state above the real axis (RA). A "left-hand above" theorem (LA) follows directly from (A.6). That is

$$f(\mathbf{k}',\mathbf{k}) = \langle \mathbf{k}|T(\Omega(\mathbf{k}) + i\eta)|\mathbf{k}'\rangle \tag{A.12}$$

Note that $\mathbf{k}$ and $\mathbf{k}'$ have been reversed in the amplitude. Similarly, Eqs. (A.5) and (A.11) give a "right-hand below" theorem (RB)

$$f^*(\mathbf{k},\mathbf{k}') = \langle \mathbf{k}|T(\Omega(\mathbf{k}') - i\eta)|\mathbf{k}'\rangle \tag{A.13}$$

and finally, the "LB" theorem becomes

$$f^*(\mathbf{k}',\mathbf{k}) = \langle \mathbf{k}|T(\Omega(\mathbf{k}) - i\eta)|\mathbf{k}'\rangle \tag{A.14}$$

APPENDIX B

Let

$$\epsilon_{1,2}^{(0)} = \frac{1}{2m}\left(\frac{1}{2}\mathbf{P} \pm \mathbf{k}\right)^2, \qquad \epsilon_{3,4}^{(0)} = \frac{1}{2m}\left(\frac{1}{2}\mathbf{P} \pm \mathbf{q}\right)^2$$

Following the prescription given in Section 3, the scattering amplitude (forward and exchange) is given to second order by

$$f_{\mathrm{P}S}(\pm\mathbf{k},\mathbf{k}) = \langle \pm\mathbf{k}|V_S|\mathbf{k}\rangle + \int \frac{d^3q}{(2\pi)^3} \frac{\langle \pm\mathbf{k}|V_S|\mathbf{q}\rangle\langle \mathbf{q}|V_S|\mathbf{k}\rangle\left[1 - f_F(\epsilon_3^{(0)}) - f_F(\epsilon_4^{(0)})\right]}{\epsilon_1^{(0)} + \epsilon_2^{(0)} - \epsilon_3^{(0)} - \epsilon_4^{(0)} + i\delta} \tag{B.1}$$

while the spectral function is

$$\langle \pm\mathbf{k}|\Gamma_S(\mathbf{P},\omega)|\mathbf{k}\rangle = \int \frac{d^3q}{(2\pi)^3}\,\delta\left[\omega - \epsilon_3^{(0)} - \epsilon_4^{(0)}\right]\langle \pm\mathbf{k}|V_S|\mathbf{q}\rangle\langle \mathbf{q}|V_S|\mathbf{k}\rangle \times\left[1 - f_F(\epsilon_3^{(0)}) - f_F(\epsilon_4^{(0)})\right] \tag{B.2}$$

Equations (3.7) and (B.2) together give

$$\begin{aligned} n(\mathbf{k}_1) = f_F(\epsilon_1) + \mathrm{P.V.}\int \frac{d^3k_2}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\sum_S D_S \\ \times \frac{\left[\langle \mathbf{k}|V_S|\mathbf{q}\rangle + (-1)^S\langle -\mathbf{k}|V_S|\mathbf{q}\rangle\right]\langle \mathbf{q}|V_S|\mathbf{k}\rangle}{\left[\epsilon_1^{(0)} + \epsilon_2^{(0)} - \epsilon_3^{(0)} - \epsilon_4^{(0)}\right]^2} \\ \times\left[1 - f_F(\epsilon_3^{(0)}) - f_F(\epsilon_4^{(0)})\right]\left[1 - f_F(\epsilon_1^{(0)}) - f_F(\epsilon_2^{(0)})\right] \\ \times\left[f_B(\epsilon_3^{(0)} + \epsilon_4^{(0)}) - f_B(\epsilon_1^{(0)} + \epsilon_1^{(0)})\right] \end{aligned} \tag{B.3}$$

With the recognition that

$$f_B[\epsilon_1^{(0)}+\epsilon_2^{(0)}]\left[1-f_F(\epsilon_1^{(0)})-f_F(\epsilon_2^{(0)})\right]=f_F(\epsilon_1^{(0)})f_F(\epsilon_2^{(0)}) \qquad \text{(B.4)}$$

so that

$$\begin{aligned}
&\left[1-f_F(\epsilon_3^{(0)})-f_F(\epsilon_4^{(0)})\right]\left[1-f_F(\epsilon_1^{(0)})-f_F(\epsilon_2^{(0)})\right]\\
&\quad\times\left[f_B(\epsilon_3^{(0)}+\epsilon_4^{(0)})-f_B(\epsilon_1^{(0)}+\epsilon_2^{(0)})\right]\\
&=f_F(\epsilon_3^{(0)})f_F(\epsilon_4^{(0)})\left[1-f_F(\epsilon_1^{(0)})\right]\left[1-f_F(\epsilon_2^{(0)})\right]\\
&\quad-f_F(\epsilon_1^{(0)})f_F(\epsilon_2^{(0)})\left[1-f_F(\epsilon_3^{(0)})\right]\left[1-f_F(\epsilon_4^{(0)})\right]
\end{aligned}$$

Equation (B.3) gives the correct second-order density distribution provided that the $\epsilon(\mathbf{k}_1)$ appearing in the first term, $f_F(\epsilon_1)$, is also given correctly to second order.

Equations (3.6), (B.1), and (B.2) together give

$$\begin{aligned}
\epsilon(\mathbf{k}_1)=\epsilon_1^{(0)}&+\int\frac{d^3k_2}{(2\pi)^3}\sum_S D_S\left[\langle\mathbf{k}|V_S|\mathbf{k}\rangle+(-1)^S\langle-\mathbf{k}|V_S|\mathbf{k}\rangle\right]f_F(\epsilon_2)\\
&+\text{P.V.}\int\frac{d^3q}{(2\pi)^3}\,\frac{\left[\langle\mathbf{k}|V_S|\mathbf{q}\rangle+(-1)^S\langle-\mathbf{k}|V_S|\mathbf{q}\rangle\right]\langle\mathbf{q}|V_S|\mathbf{k}\rangle}{\left(\epsilon_1^{(0)}+\epsilon_2^{(0)}-\epsilon_3^{(0)}-\epsilon_4^{(0)}\right)}\\
&\times\left[1-f_F(\epsilon_3^{(0)})-f_F(\epsilon_4^{(0)})\right]\left[f_B(\epsilon_3^{(0)}+\epsilon_4^{(0)})+f_F(\epsilon_2^{(0)})\right] \qquad \text{(B.5)}
\end{aligned}$$

From Eq. (B.4)

$$\begin{aligned}
&\left[1-f_F(\epsilon_3^{(0)})-f_F(\epsilon_4^{(0)})\right]\left[f_B(\epsilon_3^{(0)}+\epsilon_4^{(0)})+f_F(\epsilon_2^{(0)})\right]\\
&\quad=f_F(\epsilon_2^{(0)})\left[1-f_F(\epsilon_3^{(0)})\right]\left[1-f_F(\epsilon_4^{(0)})\right]+\left[1-f_F(\epsilon_2^{(0)})\right]f_F(\epsilon_3^{(0)})f_F(\epsilon_4^{(0)})
\end{aligned}$$

REFERENCES

1. J. M. Luttinger, *Phys. Rev.* **121**:942 (1961).
2. A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971), Chap. III; or also see C. D. Mahan, *Many-Particle Physics* (Plenum, New York, 1981), Chap. II.
3. E. Beth and G. E. Uhlenbeck, *Physica* **4**:915 (1937).
4. T. Matsubara, *Prog. Theor. Phys.* **14**:351 (1955).
5. T. Kato, T. Kobayashi, and M. Namiki, *Prog. Theor. Phys. Suppl.* **15** (1960).
6. P. C. Martin and J. Schwinger, *Phys. Rev.* **115**:1342 (1959).
7. A. Goldberg and R. D. Puff, *Phys. Rev. A* **10**:323 (1974).
8. V. M. Galitskii and A. B. Migdal, *Sov. Phys. JETP* **7**:96 (1958); V. M. Galitskii, *Sov. Phys. JETP* **7**:104 (1958); S. T. Beliaev, *Sov. Phys. JETP* **7**:299 (1958).
9. J. G. Dash, D. C. Hickernel, and E. O. McLean, *Phys. Rev. A* **8**:1589 (1973).

10. R. L. Siddon and M. Schick, *Phys. Rev. A* **9**:907 (1974); *Phys. Rev. A* **9**:1753 (1974).
11. M. B. Vetrovec and G. M. Carneiro, *Phys. Rev. B* **22**:1250 (1980).
12. R. F. Bishop, M. R. Strayer, and J. M. Irvine, *Phys. Rev. A* **10**:2423 (1974); R. F. Bishop, H. B. Ghassib, and M. R. Strayer, *Phys. Rev. A* **13**:1570 (1976); see also W. E. Brittin and A. Y. Sakakura, *Phys. Rev. A* **21**:2050 (1980).
13. R. D. Puff, *Ann. Phys. (N.Y.)* **13**:317 (1961).
14. T. Foster, *Phys. Rev.* **149**:784 (1966).
15. G. Baym and L. Kadanoff, *Phys. Rev.* **124**:287 (1961).
16. R. D. Puff, A. S. Reiner, and L. Wilets, *Phys. Rev.* **149**:778 (1966).
17. P. Nozieres, *The Theory of Interacting Fermi Systems* (W. A. Benjamin, New York, 1964), Chap. V.
18. M. Schick and R. D. Puff, unpublished.
19. B. J. Alder, private communication.